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ON GROUP PROPERTIES AND CONSERVATION LAWS FOR SECOND-ORDER QUASI-LINEAR DIFFERENTIAL EQUATIONS

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A sufficient condition for the absence of tangent transformations admitted by second-order quasi-linear differential equations and a sufficient condition for linear autonomy of operators of the Lie group of transformations admitted by second-order weakly nonlinear differential equations are found. A theorem on the structure of the first-order conservation laws for second-order weakly nonlinear differential equations is proved. A classification of second-order linear differential equations with two independent variables in terms of first-order conservation laws is proposed.

Key words: *second-order weakly nonlinear differential equations, tangent transformations, linearly autonomous operators, first-order conservation laws, Laplace invariants.*

Introduction. Group properties and conservation laws for second-order quasi-linear differential equations are studied in the paper. Information about the structure of operators admitted by a differential equation and conservation laws for the latter is known to simplify substantially both the search for these operators and conservation laws and the search for solutions of this equation [1]. Classification of differential equations in terms of conservation laws allows, in particular, identification of experimentally determined values of physical quantities and forms of dependences, which are of interest for mathematical investigation of the problem, and obtaining new physical quantities unchanged in time.

1. Tangent Transformations Admitted by Second-Order Quasi-Linear Differential Equations. We consider a second-order quasi-linear differential equation

$$a^{ij}u_{ij} + b = 0 \qquad (a^{ij} = a^{ji}), \tag{1}$$

where $u_{ij} = \partial_i\partial_j u$ and $\partial_i = \partial/\partial x^i$ $(i, j = 1, \ldots, n;\ n \geqslant 2)$; the matrix $A = \|a^{ij}\|$ and the quantity b are given functions of the variables $\boldsymbol{x} = (x^1, \ldots, x^n)$, u, and $\underset{1}{u} = (u_1, \ldots, u_n)$ $(u_i = \partial_i u)$; summation is performed over repeated indices from 1 to n (except for specially indicated cases).

For convenience, $r_*(A)$ is understood in what follows as the general rank of the matrix A [1].

Theorem 1. *If $r_*(A) \geqslant 3$, then all tangent transformations admitted by Eq.* (1) *are continued point transformations.*

Proof. Let $r_*(A) = r \geqslant 3$. Without loss of generality, we can assume that $a^{11} = -1$, two arbitrary neighboring minors in the row of minors $M^1, M^2, \ldots, M^r$ in the top left corner of the matrix A are not equal to zero, and $M^r \neq 0$. Hence, we obtain the following alternative: $M^3 \neq 0$ or $M^3 = 0$ (then, the minors M^2 and M^4 differ from zero).

The operator of the group of tangent transformations admitted by Eq. (1) is sought in the form [1]

$$\xi^i\,\partial_i + \eta\,\partial_u + \zeta_{(i)}\,\partial_{u_i}, \tag{2}$$

where $\xi^i = -H_{u_i}$, $\eta = H - u_j H_{u_j}$, $\zeta_{(i)} = H_i + u_i H_u$, and $H_i = \partial_i H$ $(i = 1, 2, \ldots, n)$; $H = H(\boldsymbol{x}, u, \underset{1}{u})$ is a generating function.

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The conditions of invariance of manifold (1) with respect to operator (2) yield the relations

$$H_{u_iu_j} = a^{ij}\lambda^1 - a^{j1}\lambda^i - a^{i1}\lambda^j \qquad (i, j = 2, 3, \ldots, n), \tag{3}$$

where $\lambda^m = H_{u_1u_m}$ $(m = 1, 2, \ldots, n)$ is the solution of the equations

$$(a^{ij}a^{k1} - a^{ik}a^{j1})\lambda^1 - (a^{ik} + a^{i1}a^{k1})\lambda^j + (a^{ij} + a^{i1}a^{j1})\lambda^k = 0; \tag{4}$$

$$\begin{aligned} 2(a^{ik}a^{mj} - a^{im}a^{jk})\lambda^1 + (a^{kj}a^{m1} - a^{mj}a^{k1})\lambda^i + (a^{im}a^{k1} - a^{ik}a^{m1})\lambda^j \\ + (a^{im}a^{j1} - a^{mj}a^{i1})\lambda^k + (a^{kj}a^{i1} - a^{ik}a^{j1})\lambda^m = 0; \end{aligned} \tag{5}$$

$$i, j, k, m = 2, 3, \ldots, n.$$

A corollary of system (4), (5) is the equations

$$B(1, i, j)(\lambda^1, \lambda^i, \lambda^j)^{\mathrm{t}} = 0 \qquad (i, j = 2, 3, \ldots, n;\ i \neq j), \tag{6}$$

where $B(1, i, j) = \|B^{km}(1, i, j)\|$ is an adjoint matrix to the third-order minor matrix of the matrix A, which is located on the intersection of rows and columns with the numbers 1, i, and j.

For the first part of the alternative, Eqs. (4)–(6) yield $\lambda^j = 0$ $(j = 1, 2, \ldots, n)$. For the second part of the alternative, Eqs. (5) yield the equations

$$M(1, i, j, k)\lambda^1 = 0 \qquad (i, j, k = 2, 3, \ldots, n; \quad i \neq j \neq k \neq i),$$

where $M(1, i, j, k)$ is the fourth-order minor of the matrix A, which is located on the intersection of rows and columns with the numbers $1, i, j, k$. It follows from here that $\lambda^1 = 0$. As $M^2 \neq 0$ and $r_*(A) \geqslant 3$, then Eq. (4) yields $\lambda^j = 0$ $(j = 2, 3, \ldots, n)$.

Thus, by virtue of Eq. (3), the generating function H is a linear function of the variables $u_1, u_2, \ldots, u_n$. Theorem 1 is proved.

Remark 1. The sufficient condition for the absence of tangent transformations for Eq. (1) $r_*(A) \geqslant 3$, generally speaking, cannot be weakened. For instance, the equation $(tu_t - u)u_{tt} = 2\nabla u \cdot \nabla u_t + 1$, where $t \in \mathbb{R}^1$, $\nabla = \partial_{\boldsymbol{x}}$, and $\boldsymbol{x} \in \mathbb{R}^{n-1}$ $(n \geqslant 2)$, admits a group of tangent transformations with a generating function $H = \exp u_t$.

2. Point Transformations Admitted by Second-Order Weakly Nonlinear Differential Equations. We consider a weakly nonlinear equation

$$a^{ij}(\boldsymbol{x}, u)u_{ij} + b(\boldsymbol{x}, u, \underset{1}{u}) = 0. \tag{7}$$

The operator admitted by Eq. (7) is sought in the form

$$\xi^i(\boldsymbol{x}, u)\partial_i + \eta(\boldsymbol{x}, u)\partial_u. \tag{8}$$

Definition 1. Operator (8) is called linearly autonomous if its coordinates $\xi^1, \xi^2, \ldots, \xi^n, \eta$ satisfy the equations $\xi^1_u = \xi^2_u = \ldots = \xi^n_u = 0$ and $\eta_{uu} = 0$.

Theorem 2. *If $r_*(A) \geqslant 2$, then the coordinates $\xi^1, \xi^2, \ldots, \xi^n$ of all operators in Eq.* (8) *admitted by the weakly nonlinear equation* (7) *are independent of the function u. If $b_{u_iu_j} = 0$ for $i, j = 1, 2, \ldots, n$ thereby, then all operators in Eq.* (8) *admitted by Eq.* (7) *are linearly autonomous.*

Proof. Let $r_*(A) = r \geqslant 2$. Without loss of generality, we can assume that $a^{11} = -1$, two arbitrary neighboring minors in the row of minors $M^1, M^2, \ldots, M^r$ in the top left corner of the matrix A are not equal to zero, and $M^r \neq 0$.

The constitutive equations contain a subsystem

$$\xi^i_u = -a^{i1}\xi^1_u, \qquad (a^{ij} + a^{i1}a^{j1})\xi^1_u = 0,$$

$$(2a^{k1}a^{ij} - a^{kj}a^{i1} - a^{ki}a^{j1})\xi^1_u = 0 \qquad (i, j, k = 1, 2, \ldots, n),$$

which implies that $\xi^i_u = 0$.

If the coordinates $\xi^1, \xi^2, \ldots, \xi^n$ of each operator in Eq. (8) admitted by Eq. (7) are independent of the function u, then we obtain

$$2\eta_{uu} = b_{u_1u_1}(\eta_u - 2a^{i1}\xi_i^1) - 2b_{u_1u_i}\xi_i^1 + \xi^1 b_{1u_1u_1} + \xi^i b_{iu_1u_1} + \eta b_{uu_1u_1}$$

$$+ b_{u_1u_1u_1}[\eta_1 + u_1(\eta_u - \xi_1^1) - u_i\xi_1^i] + b_{u_iu_1u_1}(\eta_i + u_i\eta_u - u_1\xi_i^1 - u_m\xi_i^m).$$

The second statement of the theorem follows from this equation.

Remark 2. The sufficient condition for independence of the coordinates $\xi^1, \xi^2, \dots, \xi^n$ of all operators of Eq. (8) admitted by Eq. (7) from the function u, $r_*(A) \geqslant 2$, generally speaking, cannot be weakened. For instance, the equation $u_{tt} = t|\nabla u|^2 - u_t^2$ admits the operator $(\exp u)\,\partial_t$.

Remark 3. For all operators of Eq. (8) admitted by Eq. (7) with $a^{11} \neq 0$ to be linearly autonomous, it is sufficient to require a weaker condition (than in the formulation of Theorem 2) to be satisfied, namely, $r_*(A) \geqslant 2$ and $b_{u_1u_i} = 0$ $(i = 1, 2, \dots, n)$. This condition, generally speaking, cannot be weakened. For instance, the equation $u_{tt} = \Delta u + u_t^2 - |\nabla u|^2$ admits the operator $(\exp u)\,\partial_u$.

3. First-Order Conservation Laws for Second-Order Weakly Nonlinear Differential Equations. The first-order conservation law for Eq. (7) is the vector $\boldsymbol{B} = (B^1, B^2, \dots, B^n)$ whose components are functions of the variables $\boldsymbol{x}$, u, and $\underset{1}{u}$ and satisfy the relation $(D_iB^i) = 0$ by virtue of Eq. (7) (D_i is the operator of complete differentiation with respect to x^i) [1].

Theorem 3. *If $r_*(A) \geqslant 3$, then the components $B^1, B^2, \dots, B^n$ in each first-order conservation law for Eq. (7) are polynomials of the second or lower power with respect to $u_1, u_2, \dots, u_n$.*

Proof. Let $r_*(A) = r$. Without loss of generality, we can assume that $a^{11} = -1$, and two arbitrary neighboring minors in the row of minors $M^1, M^2, \dots, M^r$ are not equal to zero.

The system of constitutive equations has the form

$$B^i_{u_j} + B^j_{u_i} + 2a^{ij}B^1_{u_1} = 0 \quad (i, j = 1, 2, \dots, n), \qquad B^i_i + u_iB^i_u + bB^1_{u_1} = 0.$$

The conditions of compatibility of this system are the equations $B^k_{u_iu_j} = a^{ij}B^1_{u_1u_k} - a^{ik}B^1_{u_1u_j} - a^{jk}B^1_{u_1u_i}$ and Eqs. (4) and (5) where it should be assumed that $\lambda^k = B^1_{u_1u_1u_k}$ $(i, j = 2, 3, \dots, n;\ k = 1, 2, \dots, n)$. Therefore, $B^1_{u_1u_1u_k} = 0$ for all $k = 1, 2, \dots, n$. The statement of the theorem follows from here.

4. First-Order Conservation Laws for Second-Order Linear Differential Equations. We consider an equation

$$L[u] = a^{ij}(\boldsymbol{x})u_{ij} + b^i(\boldsymbol{x})u_i + c(\boldsymbol{x})u = 0. \tag{9}$$

It follows from Green's operator formula

$$vL[u] - uL^*[v] = D_i[a^{ij}vu_j + (b^iv - (va^{ij})_j)u]$$

that Eq. (9) has the conservation law

$$B^i = a^{ij}vu_j + (b^iv - (va^{ij})_j)u \qquad (i = 1, 2, \dots, n), \tag{10}$$

where $v = v(\boldsymbol{x})$ is the solution of the adjoint equation $L^*[v] = 0$.

Definition 2. If the conservation law for Eq. (9) is a linear combination of trivial conservation laws and conservation laws (10), it is called an obvious conservation law; otherwise, it is called a non-obvious conservation law.

Theorem 3 gives an "estimate from above" for non-obvious first-order conservation laws for Eq. (9). The "estimate from below" is obtained from

Theorem 4. *Any conservation law whose components are linearly expressed via $u, u_1, u_2, \dots, u_n$ is an obvious conservation law for Eq. (9).*

Proof. Without loss of generality, we can assume that $a^{11} = -1$. Let the conservation law have the form

$$B^i = f^{ij}(\boldsymbol{x})u_j + h^i(\boldsymbol{x})u \qquad (i = 1, 2, \dots, n).$$

It follows from the relation $(D_iB^i) = 0$ by virtue of Eq. (9) that

$$f^{ij} = -a^{ij}v + g^{ij}(\boldsymbol{x}), \qquad g^{ij} + g^{ji} = 0 \qquad (i, j = 1, 2, \dots, n),$$

where $L^*[v] = 0$. To finalize the proof, it is sufficient to note that the matrix $\|g^{ij}\|$ is antisymmetric.

5. Classification of Second-Order Linear Differential Equations with Two Independent Variables in Terms of First-Order Conservation Laws. Below we perform a classification of the equation

$$u_{12} + au_1 + bu_2 + c = 0 \tag{11}$$

with respect to the coefficients a, b, and c, which are given functions of the variables x^1 and x^2, in terms of the first-order conservation laws, i.e., in terms of the vectors $\boldsymbol{B} = (B^1, B^2)$ whose components depend on x^1, x^2, u, u_1, and u_2 and satisfy the relation

$$(D_1 B^1 + D_2 B^2) = 0 \tag{12}$$

by virtue of Eq. (11).

The most general equivalence transformations, which retain the differential structure of Eq. (11) and relation (12), consist of the transformations $x'^1 = f^1(x^1)$, $x'^2 = f^2(x^2)$, and $u = g(x^1, x^2)u'$, where $f^1(x^1)$ and $f^2(x^2)$ are reversible functions; $g(x^1, x^2) \neq 0$.

The system of constitutive equations is brought to the form

$$\begin{gathered} B^1_{u_1} = B^2_{u_2} = 0, \\ B^1_1 + u_1 B^1_u + B^2_2 + u_2 B^2_u - (au_1 + bu_2 + cu)(B^1_{u_2} + B^2_{u_1}) = 0. \end{gathered} \tag{13}$$

If a, b, and c are arbitrary functions of the variables x^1 and x^2, then the set of the first-order conservation laws for Eq. (11) consists only of trivial conservation laws.

The classification is performed in terms of non-obvious conservation laws. The second continuation of system (13) contains the classifying equations $hB^1_{u_2u_2u_2} = 0$ and $kB^2_{u_1u_1u_1} = 0$, where $h = a_1 + ab - c$ and $k = b_2 + ab - c$ are the Laplace invariants.

5.1. *Case with $hk \neq 0$.* It follows from the second continuation of system (13) that the components B^1 and B^2 have the form

$$\begin{aligned} B^1 &= \theta^1 u_2^2 + \psi u u_2 - (\psi_2/2 - a\psi - \theta^2 c)u^2, \\ B^2 &= \theta^2 u_1^2 + \psi u u_1 - (\psi_1/2 - b\psi - \theta^1 c)u^2, \end{aligned} \tag{14}$$

where $\psi = \theta^1 a + \theta^2 b$, $\theta^1 = \varphi^1/h$, and $\theta^2 = \varphi^2/k$; the functions $\varphi^1 = \varphi^1(x^1, x^2)$ and $\varphi^2 = \varphi^2(x^1, x^2)$ are the solution of the overdetermined passive system:

$$\begin{gathered} \varphi^1_1 = (p_1 + 2b)\varphi^1, \qquad \varphi^2_2 = (q_2 + 2a)\varphi^2, \qquad \varphi^1_2 + \varphi^2_1 = 2(a\varphi^1 + b\varphi^2), \\ \varphi^1_{22} = (q_2 + 4a)\varphi^1_2 + 2[a_2 - a(q_2 + 2a)]\varphi^1 - (h - K)\varphi^2, \\ \varphi^1_{11} = (p_1 + 4b)\varphi^2_1 + 2[b_1 - b(p_1 + 2b)]\varphi^2 - (k - H)\varphi^1, \\ 2(K - H)\varphi^1_2 - [H_2 - q_2 H + 4a(K - H)]\varphi^1 = (K_1 - p_1 K)\varphi^2; \end{gathered} \tag{15}$$

$p = \ln h$; $q = \ln k$; $H = 2h - k - p_{12}$ is the Laplace invariant of the Laplace x^1-transformation of Eq. (11); $K = 2k - h - q_{12}$ is the Laplace invariant of the Laplace x^2-transformation of Eq. (11). The solution of system (15) has the greatest arbitrariness if the last equation of this system is an identity. If the last equation of system (15) is satisfied identically, then it follows from the remaining equations of this system that all coefficients of the expansions of φ^1 and φ^2 into the Taylor series are expressed via no more than three arbitrary constants. In the case with $kh \neq 0$, therefore, Eq. (11) has no more than three non-obvious first-order conservation laws.

The greatest number of non-obvious first-order conservation laws is obtained for those equations (11) whose coefficients a, b, and c satisfy the system of equations

$$K = H, \qquad H_1 = p_1 H, \qquad H_2 = q_2 H \tag{16}$$

with the compatibility condition $(k - h)H = 0$. This equation is a classifying equation.

Let $k = h$. It follows from Eqs. (16) that h is the solution of the Liouville equation $(\ln h)_{12} = \gamma h$ ($\gamma = \text{const}$). Then, Eq. (11) with $\gamma \neq 0$ is equivalent (see [1]) to the Euler–Poisson equation

$$u_{12} - \frac{2}{\gamma(x^1 + x^2)}(u_1 + u_2) + \frac{4u}{\gamma^2(x^1 + x^2)^2} = 0, \tag{17}$$

and Eq. (11) with $\gamma = 0$ is equivalent to the equation

$$u_{12} + x^1 u_1 + x^2 u_2 + x^1 x^2 u = 0. \tag{18}$$

It follows from the solution of system (15) that each of Eqs. (17) and (18) has three non-obvious first-order conservation laws determined by Eqs. (14), where, respectively,

$$\theta^1 = \frac{c^1 + c^2 x^2 + c^3 (x^2)^2}{(x^1 + x^2)^{4/\gamma}}, \qquad \theta^2 = \frac{-c^1 + c^2 x^1 - c^3 (x^1)^2}{(x^1 + x^2)^{4/\gamma}}; \tag{19}$$

$$\theta^1 = (c^1 + c^2 x^2) \exp(2x^1 x^2), \qquad \theta^2 = (c^3 - c^2 x^1) \exp(2x^1 x^2). \tag{20}$$

Let $k \neq h$. Then we have $H = K = 0$. In this case, Eq. (11) has three non-obvious first-order conservation laws determined by Eqs. (14), where

$$\theta^1 = y(x^2) \exp\Big(2 \int b\, dx^1\Big), \qquad \theta^2 = z(x^1) \exp\Big(2 \int a\, dx^2\Big). \tag{21}$$

The function $y(x^2)$ is the solution of the third-order linear ordinary differential equation, and the function $z(x^1)$ is expressed via $y(x^2)$ and the coefficients of Eq. (11).

The results of the classification with $kh \neq 0$ are formulated as the following theorem.

Theorem 5. *For $kh \neq 0$, Eq.* (11) *has no more than three non-obvious first-order conservation laws where the components are quadratic dependences of u, u_1, and u_2.*

For $h = k$, Eq. (11) *has three non-obvious first-order conservation laws if and only if it is equivalent either to the Euler–Poisson equation* (17) *or to Eq.* (18) *for which non-obvious first-order conservation laws are determined by Eqs.* (14), (19), *and* (20).

For $h \neq k$, Eq. (11) *has three non-obvious first-order conservation laws if and only if $H = K = 0$. In this case, non-obvious first-order conservation laws are determined by Eqs.* (14) *and* (21).

5.2. *Case with $kh = 0$.* The case with $h = k = 0$ is trivial. If one of the invariants h or k differs from zero, then we can assume that $h \neq 0$ without loss of generality. The second continuation of system (13) contains the classifying equation $[(\ln h)_{12} - 2h] B^1_{u_2 u_2} = 0$. For $(\ln h)_{12} \neq 2h$, the non-obvious first-order conservation laws for Eq. (11) have the form $B^1 = 0$ and $B^2 = f(x^1, v)$, where $v = (u_1 + bu) \exp\Big(\int a\, dx^2\Big)$; f is an arbitrary function ($f_v \neq 0$). For $(\ln h)_{12} = 2h$, Eq. (11) is equivalent to the Euler–Poisson equation [1] $u_{12} - u_1/(x^1 + x^2) = 0$ for which there is one more non-obvious conservation law

$$B^1 = u_2^2 - \frac{u u_2}{x^1 + x^2} + \frac{u^2}{2(x^1 + x^2)^2}, \qquad B^2 = -\frac{u u_1}{x^1 + x^2} - \frac{u^2}{2(x^1 + x^2)^2}.$$

5.3. *Invariant Description of the Cases with the Maximum Extension of the Set of Conservation Laws.* It was found above that Eq. (11) with $hk \neq 0$ has three non-obvious first-order conservation laws if and only if the Laplace invariants h and k satisfy either the conditions

$$p_{12} = 2 \exp p - \exp q, \qquad q_{12} = 2 \exp q - \exp p \qquad (p \neq q) \tag{22}$$

or the conditions

$$p_{12} = \gamma \exp p \qquad (\gamma = \text{const}, \quad p = q). \tag{23}$$

For $k = 0$ and $h \neq 0$, extension of the set of non-obvious first-order conservation laws for Eq. (11) in the nontrivial case occurs only if the Laplace invariants of this equation satisfy the condition

$$p_{12} = 2 \exp p \qquad (k = 0, \quad h \neq 0). \tag{24}$$

The solution of the system of constitutive equations shows that the main groups of Eqs. (22)–(24) coincide with the group of equivalence transformations of Eq. (11). The basis of the second-order differential invariants of this group is found by a known algorithm [1]. This basis consists of six functions, which can be chosen in the following manner:

$$I^1 = \frac{k}{h}, \quad I^2 = \frac{1}{h} (\ln h)_{12}, \quad I^3 = \frac{1}{k} (\ln k)_{12},$$

$$I^4 = \frac{1}{I_1^1}\Big(\ln\frac{I_1^1}{h}\Big), \quad I^5 = \frac{1}{I_2^1}\Big(\ln\frac{I_2^1}{h}\Big), \quad I^6 = \frac{1}{h}\, I_1^1 I_2^1.$$

The invariants I^1, I^2, and I^3 were first found by Ovsyannikov [2]. With the use of these invariants, Eqs. (22), (23), and (24) are written in the following form, respectively:

$$I^2 + I^1 = 2, \qquad I^3 + \frac{1}{I^1} = 2 \qquad (I^1 \neq 0; 1); \tag{25}$$

$$I^1 = 1, \qquad I^2 = \gamma = \text{const}; \tag{26}$$

$$I^1 = 0, \qquad I^2 = 2. \tag{27}$$

Thus, we proved the following theorem.

Theorem 6. *The non-trivial maximum extension of the set of first-order conservation laws for Eq.* (11) *with $h \neq 0$ is observed if and only if the Ovsyannikov invariants I^1, I^2, and I^3 of this equation satisfy the alternative relations* (25), (26), *and* (27), *respectively.*

Theorem 6 gives an invariant description of Eqs. (11) with the maximum number of non-obvious first-order conservation laws.

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